

Quark-Antiquark Bound States and the Breit Equation

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Abstract

A non-covariant but approximately relativistic two-body wave equation (Breit equation) describing the quantum mechanics of two fermions interacting with one another through a potential containing scalar, pseudoscalar and vector parts is presented. After expressing the sixteen component two-body wavefunction in terms of a radial and an angular function by means of the multipole expansion, the initial equation can be reduced into a set of sixteen radial equations which, in turn, can be classified in accordance to the parity and the state of the wavefunctions involved. The adequacy of the reduced equations in describing real problems is discussed by applying the theory to QCD problems and the calculation of the energies of bound states of quark-antiquark systems is performed to order α^4 . We show that bound states of heavy quarks can be described adequately by the Breit equation for a funnel interaction between the particles.

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1 Introduction

Over the years a number of equations has been introduced in an effort to describe adequately the relativistic dynamics of a system of two interacting spin- $\frac{1}{2}$ particles ([1]-[6]). The problem is to determine how the two fermions behave when they are influenced by their mutual interaction. The simplest relativistic equation for fermions, the Dirac equation, which can describe the quantum mechanics of a single fermion, is not useful for a two-fermion system unless the mass of one of the particles is much larger than that of the other particle. One equation that has been extensively used, in the past, is the equation Breit proposed in 1929, which describes the interaction between two electrons [2, 6, 7, 8]

$$[E - H_{(1)} - H_{(2)} - V_{int}(|\vec{x}_1 - \vec{x}_2|)]\psi(\vec{x}_1, \vec{x}_2) = 0 \quad (1)$$

where $H_{(i)} = \vec{\alpha}_i \cdot \vec{p}_i + \beta m$ ($i=1,2$) is identical with the Dirac Hamiltonian of the i particle and E is the total energy of the system. The interaction between the two particles equals $V_{int} = V_c + V_B$ and it is partly due to an *instantaneous* (static) Coulomb interaction V_c and partly due to effects prescribed by quantum electrodynamics described by *retardation* terms

$$V_B(\vec{r}) = -\frac{1}{2}V_c(\vec{r})(\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \vec{\alpha}_1 \cdot \hat{r}\vec{\alpha}_2 \cdot \hat{r})$$

which is referred to, as “Breit Interaction”. $\vec{\alpha}_i$ are the Dirac matrices, $\vec{r}(\equiv \vec{x}_1 - \vec{x}_2)$ is the relative position vector and \hat{r} is the unit vector.

Although the initial Breit equation involved two electrons [7], we aim to generalise it for any two fermions. There are two requirements that the Eq.(1) clearly satisfies: (i) In the limit of negligent interaction between the particles, Eq.(1) implies that the total energy of the system equals the sum of the energies of the particles and the stationary wavefunction $\psi(\vec{x}_1, \vec{x}_2)$ is the product of the wavefunctions $\psi(\vec{x}_1)$, $\psi(\vec{x}_2)$ which are simply the solutions of the Lorentz-invariant Dirac equations for each particle, separately, (ii) In the non-relativistic limit, the Breit equation reduces to the Schrödinger equation for a two-particle system.

If retardation effects are ignored, the Breit equation reduces to the Dirac equation in the infinite limit of the mass of one of the constituent particles and then, either equation can be used. Certainly, the former would provide a more accurate description because corrections due to the motion of the heavier particle should be taken into account.

Unlike the Dirac equation, the Breit equation lacks a very important ingredient characterising the relativistic equations, the covariance. Eq.(1) is not written in a covariant notation since individual terms in the part representing the interaction are not Lorentz invariant (the potential is not a relativistically invariant quantity). Besides this, not only does the wavefunction in the Breit equation depend on the positions \vec{x}_1 and \vec{x}_2 but also depends on one time-variable (rather than an one time-variable each for the two particles), a fact that does not allow a Lorentz-invariant formulation. However, it can be considered covariant in the centre-of-mass-frame because, in that case, the relative motion of the particles is studied and there is only one position vector which comes into the equation, namely that one which measures the distance between the two electrons, \vec{r} . Attempts have been made aiming at deriving a covariant Breit equation by means of constraining the equations. It is too early to say whether this work is successful [9].

In spite of the fact that Breit equation manifestly lacks covariance, it appears that it can describe efficiently the quantum mechanics of some two-body systems (at least approximately to the required accuracy) for long-range interactions with small coupling [7, 10]. When the equation was proposed by Breit in 1929 [2], QED was known and the long range Coulomb type potential $V_c(\vec{r})$ was the form of the instantaneous interaction for hydrogen-like atoms. In contrast to the static $V_c(\vec{r})$ which is the zero-order term in the (v/c) expansion of the electromagnetic interaction, the Breit interaction $V_B(\vec{r})$ is the $(v/c)^2$ term (velocity-dependent) and it constitutes only an approximation to the relativistic interaction between the two particles. The study of the fine and hyperfine structure for hydrogen-like atoms indicates that the Breit interaction should not be considered on the same footing as the Coulomb potential $V_c(\vec{r})$. On the contrary, it should be treated as a small perturbation, otherwise it does not lead to correct results ([10]-[15]). The fact that the Breit equation gives satisfactory results for the Coulomb potential implies that the equation, although not covariant, can provide a good description to a two-body system for long-range interactions.

In this work, we aim to extend the application of the Breit equation to systems of fermions that interact with one another through a short-range strong potential. Due to the fact that the distance between the two particles is very small, one position vector suffices to describe the system which puts in the same footing time and position. Then, one might consider the Breit equation is approximately covariant and more compatible

with the special theory of relativity, in the case of short-range strong interactions.

There has been a belief that the internal dynamics of quark-antiquark systems can be described satisfactorily by two-body semi-relativistic equations and some attempts have been made towards this direction ([13],[16]-[34]). This is a very interesting possibility since it suggests a $q\bar{q}$ bound state could provide a sort of “Hydrogen-atom” for QCD. In this paper, we consider fermionia composed from heavy quarks (bottom, charm) and we assume that the particles interact with each other through a “funnel” potential [16, 28]. Throughout this paper, we will handle the Breit equation by taking into account the instantaneous interaction between the particles and treating the non-static terms perturbatively. It turns out that the Breit equation offers a very good description of the systems and this is partly due to the fact that the constituents are heavy particles whose speed are very small so as to declare that a semi-relativistic treatment gives satisfactory results for certain bound states of the two fermions. Unfortunately, this does not seem to hold true in the event of higher bound states as well as for bound states of lighter particles because a relativistic treatment is necessary.

One of the advantages of the Breit-equation is that it can be handled relatively easily by means of some standard techniques (i.e. multipole technique [23, 35, 36, 37, 38]). The wavefunction describing the system is a sixteen component function and it can be used to obtain radial equations.

This paper is organised as follows. We start in Section 2 with the introduction of the Breit equation and we assume a general interaction between two spin-1/2 particles including a vector, scalar and pseudoscalar particle exchange. Using the multipole technique, we separate the equation into its radial and angular parts and derive sixteen radial equations grouped according to the values of the spin of the system. Then, we present the non-relativistic type of the potential which describes the interaction between two quarks [Section 3] and we derive the radial equations that apply in that case [Section 4]. In Section 5, we solve the equations for two types of quarkonia, bottomium and charmonium, and we calculate the energy levels of various bound states. This consideration allows us to check the validity of the Breit equation by comparing the results derived from the theory with well-established results for bottomium and charmonium. Section 6 contains brief concluding remarks.

2 The Breit equation and its reduction to radial equations

We consider a two-body Dirac equation (Breit equation) [2, 6, 7, 8]

$$\left[E - \gamma_0^{(1)}(\vec{\gamma}^{(1)} \cdot \vec{p} + m_1) - \gamma_0^{(2)}(-\vec{\gamma}^{(2)} \cdot \vec{p} + m_2) - V_{int}(|\vec{r}|) \right] \psi(\vec{r}) = 0 \quad (2)$$

describing a system of two spin-1/2 particles of masses m_1 and m_2 , in the centre of mass frame, interacting with each other through a static central potential of the form [10, 38]

$$\begin{aligned} V_{int}(r) &= V_S(r) + V_P(r) + V_V(r) \\ V_S(r) &= -(\gamma_0^{(1)} \otimes \gamma_0^{(2)})S(r) \\ V_P(r) &= -(i\gamma_0^{(1)}\gamma_5^{(1)}) \otimes (i\gamma_0^{(2)}\gamma_5^{(2)})P(r) \\ V_V(r) &= +[(\gamma_0^{(1)}\gamma_\mu^{(1)}) \otimes (\gamma_0^{(2)}\gamma_\mu^{(2)})]_{\mu=0} V(r) \end{aligned} \quad (3)$$

where $\vec{r} \equiv \vec{x}_1 - \vec{x}_2$, $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$, $r \equiv |\vec{x}_1 - \vec{x}_2|$, and $V_S(r)$, $V_P(r)$ and $V_V(r)$ are the parts of the interaction with scalar, pseudoscalar and vector Lorentz structure, respectively. The choice of the combination of the γ matrices which leads to the appropriate Lorentz structure is not unique, however, the selection is based on some conditions which should be satisfied. To be more specific, $\gamma_0^{(1)} + \gamma_0^{(2)}$ has a scalar Lorentz structure, as well, and although it seems reasonable because it couples the potential $S(r)$ directly to the mass of each particle (a feature of the scalar potential in contrast to the vector potential which couples to the charge of the particles) does not flip the helicities of the fermions in the case of the chiral representation.

The signs in front of the various forms of the potentials can be justified as follows: in QED, the vector exchange is actually the Coulomb interaction between particles and if their charges have the same sign, then $V_V(r)$ is positive. On the other hand, for scalar and pseudoscalar exchange the sign of propagator is opposite to the Coulomb part of the photon propagator.

The superscript i ($i = 1, 2$) which appears in the γ matrices refers to the particle. The reason why in the case of potentials with vector Lorentz structure only the $\gamma_0^{(1)}$, $\gamma_0^{(2)}$ contributions have been taken into account, is that the $\vec{\gamma}^{(i)}$ matrices give a non-static character to the potential since they introduce velocity terms ¹. $\gamma^{(1)}(\gamma^{(2)})$ is the Dirac matrix γ

¹We recall the relation $\vec{\gamma}^{(i)} = \gamma_0^{(i)} \vec{\alpha}^{(i)} = \gamma_0^{(i)} \frac{\vec{v}^{(i)}}{c}$, where $\vec{v}^{(i)}$ is the velocity of the i particle.

acting in the subspace of the spinor of particle 1 (2) and it acts on ψ from the left (right)

$$\begin{aligned}\gamma^{(1)}\psi &\equiv \gamma^{(1)}\psi \\ \gamma^{(2)}\psi &\equiv \psi(\gamma^{(2)})^T\end{aligned}\quad (4)$$

The total spin of the system of the two particles is either $S = 0$ or $S = 1$, therefore the total angular momentum of the system is either $j = l$ (for $S = 0$) or $j = l \pm 1, l$ (for $S = 1$).

The form Eq.(2) acquires depends on the representation of the γ matrices. In the next subsection, Eq.(2) is written in the Dirac-Pauli representation, where $\gamma_0^{(i)}$ matrices are taken to be diagonal. This choice will allow us to examine the connection with the non-relativistic limit.

2.1 Dirac representation

In the Dirac representation, $\gamma_0^{(i)}$ matrix is diagonal

$$\gamma_0 = \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\quad (5)$$

The spinor $\psi(\vec{r})$ is a sixteen-component wave function and it can be represented as a 4×4 matrix

$$\psi(\vec{r}) = (\psi_{\gamma_0^{(1)}\gamma_0^{(2)}}) = \begin{pmatrix} \psi_{++} & \psi_{+-} \\ \psi_{-+} & \psi_{--} \end{pmatrix}\quad (6)$$

where the indices $+, -$ are the eigenvalues $(+1, -1)$ of the Dirac matrices $\gamma_0^{(1)}, \gamma_0^{(2)}$ in the so-called double Dirac representation [10, 36, 37]. The left index refers to the first particle and the right one to the second particle. By inserting (4) and (5) and (6) into Eq.(2), the Breit equation takes the form

$$\begin{aligned}E &\begin{pmatrix} \psi_{++} & \psi_{+-} \\ \psi_{-+} & \psi_{--} \end{pmatrix} - \begin{pmatrix} \vec{\sigma}^{(1)} \cdot \vec{p}\psi_{-+} & \vec{\sigma}^{(1)} \cdot \vec{p}\psi_{--} \\ \vec{\sigma}^{(1)} \cdot \vec{p}\psi_{++} & \vec{\sigma}^{(1)} \cdot \vec{p}\psi_{+-} \end{pmatrix} - m_1 \begin{pmatrix} \psi_{++} & \psi_{+-} \\ -\psi_{-+} & -\psi_{--} \end{pmatrix} + \\ &+ \begin{pmatrix} \vec{\sigma}^{(2)} \cdot \vec{p}\psi_{+-} & \vec{\sigma}^{(2)} \cdot \vec{p}\psi_{++} \\ \vec{\sigma}^{(2)} \cdot \vec{p}\psi_{--} & \vec{\sigma}^{(2)} \cdot \vec{p}\psi_{-+} \end{pmatrix} - m_2 \begin{pmatrix} \psi_{++} & -\psi_{+-} \\ \psi_{-+} & -\psi_{--} \end{pmatrix} + S(r) \begin{pmatrix} \psi_{++} & -\psi_{+-} \\ -\psi_{-+} & \psi_{--} \end{pmatrix} - \\ &- P(r) \begin{pmatrix} \psi_{--} & -\psi_{-+} \\ -\psi_{+-} & \psi_{++} \end{pmatrix} - V(r) \begin{pmatrix} \psi_{++} & \psi_{+-} \\ \psi_{-+} & \psi_{--} \end{pmatrix} = 0\end{aligned}\quad (7)$$

In order to simplify Eq.(7) and bring it to a form which can be handled easily, we introduce the following components [36, 37]

$$\begin{aligned}\left. \begin{matrix} \phi \\ \phi^0 \end{matrix} \right\} &= P_0 \frac{i}{\sqrt{2}} (\psi_{++} \mp \psi_{--}) \\ \left. \begin{matrix} \vec{\phi} \\ \vec{\phi}^0 \end{matrix} \right\} &= \frac{1}{2} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) P_1 \frac{1}{\sqrt{2}} (\psi_{+-} \pm \psi_{-+})\end{aligned}\quad (8)$$

$$\left. \begin{array}{l} \chi \\ \chi^0 \\ \vec{\chi} \\ \vec{\chi}^0 \end{array} \right\} = \begin{array}{l} P_0 \frac{i}{\sqrt{2}} (\psi_{+-} \mp \psi_{-+}) \\ \frac{1}{2} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) P_1 \frac{1}{\sqrt{2}} (\psi_{++} \pm \psi_{--}) \end{array} \quad (9)$$

where

$$\begin{aligned} P_0 &= \frac{1}{4} \left(1 - \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \right) \\ P_1 &= \frac{1}{4} \left(3 + \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \right) \end{aligned} \quad (10)$$

are the projection operators on states with total spin $S = 0$ and $S = 1$, respectively,

$$\begin{aligned} P_0 |\text{state}\rangle &= |\text{state } S = 0\rangle \\ P_1 |\text{state}\rangle &= |\text{state } S = 1\rangle \end{aligned} \quad (11)$$

The components $\phi, \phi^0, \chi, \chi^0$ and $\vec{\phi}, \vec{\phi}^0, \vec{\chi}, \vec{\chi}^0$ correspond to spin $S = 0$ and $S = 1$, respectively and they are functions of \vec{r} . Noting that

$$\begin{aligned} P_0 (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) &= (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) P_1 \\ P_0 (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) &= (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) P_0 = 0 \end{aligned} \quad (12)$$

and by means of the identities

$$\begin{aligned} P_0 (\sigma_i^{(1)} \sigma_k^{(2)} + \sigma_k^{(1)} \sigma_i^{(2)}) &= -2\delta_{ik} P_0 \\ P_0 (\sigma_i^{(1)} \sigma_k^{(2)} - \sigma_k^{(1)} \sigma_i^{(2)}) &= iP_0 \epsilon_{ikl} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})_l \\ \hookrightarrow P_0 (\sigma_i^{(1)} - \sigma_i^{(2)}) (\sigma_k^{(1)} \pm \sigma_k^{(2)}) &= \begin{cases} 2iP_0 \epsilon_{ikl} (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})_l \\ 2P_0 \delta_{ik} \end{cases} \end{aligned} \quad (13)$$

Eq.(7) leads to the following set of component wave equations [10, 36, 37]

$$\begin{aligned} \frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \phi^0 - \frac{(m_1+m_2)}{2} \phi - i\vec{p} \cdot \vec{\phi} &= 0 \\ \frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \phi - \frac{(m_1+m_2)}{2} \phi^0 &= 0 \\ \frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \chi^0 - \frac{(m_1-m_2)}{2} \chi - i\vec{p} \cdot \vec{\chi} &= 0 \\ \frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \chi - \frac{(m_1-m_2)}{2} \chi^0 &= 0 \\ \frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \vec{\chi} - \frac{(m_1+m_2)}{2} \vec{\chi}^0 + i\vec{p} \chi^0 &= 0 \\ \frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \vec{\chi}^0 - \frac{(m_1+m_2)}{2} \vec{\chi} + i\vec{p} \times \vec{\phi}^0 &= 0 \\ \frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \vec{\phi} - \frac{(m_1-m_2)}{2} \vec{\phi}^0 + i\vec{p} \phi^0 &= 0 \\ \frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \vec{\phi}^0 - \frac{(m_1-m_2)}{2} \vec{\phi} + i\vec{p} \times \vec{\chi}^0 &= 0 \end{aligned} \quad (14)$$

The type of the potential we consider is central therefore the next step we will follow is the introduction of a method that will eventually separate the angular from the radial dependences as it was performed in the three dimensional Schrödinger equation. To this end, it is convenient to introduce the derivative

$$\frac{\partial}{\partial n_{i\perp}} \equiv (\delta_{ik} - n_i n_k) \frac{\partial}{\partial n_k} \quad (15)$$

where $\vec{n} = \frac{\vec{r}}{r}$ is a three dimensional vector, while $\frac{\partial}{\partial \vec{n}_\perp}$ lies in the perpendicular plane $\left[n_i \frac{\partial}{\partial n_{i\perp}} = n_i (\delta_{ik} - n_i n_k) \frac{\partial}{\partial n_k} = n_k \frac{\partial}{\partial n_k} - n_k \frac{\partial}{\partial n_k} = 0 \right]$. Some useful identities that are going to be used are the following

$$\begin{aligned} \frac{\partial}{\partial \vec{r}} &= \frac{1}{r} \frac{\partial}{\partial \vec{n}_\perp} + \vec{n} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \vec{n}_\perp} \cdot \vec{n} &= (\delta_{ik} - n_i n_k) \frac{\partial}{\partial n_k} n_i = (\delta_{ik} - n_i n_k) \delta_{ki} = 3 - 1 = 2 \end{aligned} \quad (16)$$

Then, $\vec{\nabla}^2 = \frac{\partial^2}{\partial \vec{r}^2} = \left(\frac{1}{r} \frac{\partial}{\partial \vec{n}_\perp} + \vec{n} \frac{\partial}{\partial r} \right)^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \vec{n}_\perp^2} + \frac{2}{r} \frac{\partial}{\partial r}$. This form of the Laplacian indicates that $\frac{\partial^2}{\partial \vec{n}_\perp^2}$ was correctly regarded as the angular part of the Laplacian.

2.2 Expansion of the wavefunctions

The component functions defined by (8) and (9) are classified in two groups: the scalar $(\phi, \phi^0, \chi, \chi^0)$ which refer to states with $S = 0$, and the vector $(\vec{\phi}, \vec{\phi}^0, \vec{\chi}, \vec{\chi}^0)$ which correspond to states with $S = 1$ [10, 36, 37, 38]. These functions can be expanded in the following way:

(i). Scalar functions ($S = 0$).

The scalar functions can be written as

$$B(\vec{r}) = \sum_{j,m} B(r) Y_{jm}(\vec{n}) \quad (17)$$

where $B(\vec{r})$ stands for any of the four $S = 0$ component functions, $B(r)$ is the radial part of the function and $Y_{jm}(\vec{n})$ are the spherical harmonics depending only on the angles.

(ii). Vector functions ($S = 1$).

We define an operator S_k which acts on any of the four vector components [(8), (9)] as follows

$$S_k(\vec{A}(\vec{r}))|_i \equiv \frac{1}{2}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})_i P_1 \left(\frac{1}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})_k \right) \frac{1}{\sqrt{2}}(\dots) \quad (18)$$

where (...) equals $(\psi_{+-} + \psi_{-+})$ (for $\vec{\phi}$), $(\psi_{+-} - \psi_{-+})$ (for $\vec{\phi}^0$), $(\psi_{++} + \psi_{--})$ (for $\vec{\chi}$), $(\psi_{++} - \psi_{--})$ (for $\vec{\chi}^0$) and $\vec{A}(\vec{r})$ is any of the four vector components. We notice that the operator S_k does not act on $\vec{A}(\vec{r})$ from the left since in that case it would give zero $\left(\frac{1}{2}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})_k \frac{1}{2}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})_i P_1 \stackrel{(12)}{=} 0\right)$. From the definition of $S_k(\vec{A}(\vec{r}))|_i$, it is clear that $S_k(\vec{A}(\vec{r}))|_i = i\epsilon_{ikl} \phi_l$. By making use of the multipole technique [10, 36, 37], it is possible to expand any vector component $\vec{A}(\vec{r})$ into three parts: “electric” (A_e), “longitudinal” (A_l) and “magnetic” (A_m) defined as

$$\begin{aligned} A_e(\vec{r}) &= \sum_{j,m} A_e(r) Y_{jm}(\vec{n}) \\ A_l(\vec{r}) &= \sum_{j,m} A_l(r) Y_{jm}(\vec{n}) \\ A_m(\vec{r}) &= \sum_{j,m} A_m(r) Y_{jm}(\vec{n}) \end{aligned}$$

$$\vec{A}(\vec{r}) = \underbrace{\vec{n} A_e(\vec{r})}_{\vec{A}_e(\vec{r})} - \underbrace{\frac{\partial}{\partial \vec{n}_\perp} \frac{A_l(\vec{r})}{j(j+1)}}_{\vec{A}_l(\vec{r})} - \underbrace{(\vec{n} \times \frac{\partial}{\partial \vec{n}_\perp}) \frac{A_m(\vec{r})}{j(j+1)}}_{\vec{A}_m(\vec{r})} \quad (19)$$

(In the rest of our study, for the sake of simplicity, we will drop the summation symbol $\sum_{j,m}$). It may seem that j in Y_{jm} is the orbital momentum rather than the total one. But this is not true for the function $\vec{A}(\vec{r})$ because the vectors to which it is proportional, depend on angles. Actually, if \vec{S} and \vec{L} are the total spin and total orbital angular momentum operators, respectively, then for each part of $\vec{A}(\vec{r})$, for instance $\vec{A}_e(\vec{r}) \equiv \vec{n} A_e(r) Y_{jm}(\vec{n})$, one has [10]

$$\begin{aligned} J_k(\vec{A}_e(\vec{r}))|_i &= S_k(\vec{A}_e(\vec{r}))|_i + L_k(\vec{A}_e(\vec{r}))|_i = \\ &= i\epsilon_{ikl} n_l A_e(r) Y_{jm}(\vec{n}) - i\epsilon_{kla} n_l \frac{\partial}{\partial n_a} (n_i A_e(r) Y_{jm}(\vec{n})) = \\ &= -i\epsilon_{kla} n_l n_i A_e(r) \frac{\partial Y_{jm}(\vec{n})}{\partial n_a} = \vec{n}_i L_k(A_e(r) Y_{jm}(\vec{n})) \implies \\ \implies \vec{J}^2 \vec{A}_e(\vec{r}) &= \vec{n} L^2(A_e(r) Y_{jm}(\vec{n})) = j(j+1) \vec{A}_e(\vec{r}) \end{aligned}$$

which implies that j is the total angular momentum. By inserting [(17),(19)] into Eqs.(14), we manage to eliminate the angular dependences and the Breit equation reduces to the following set of sixteen radial equations

$$\begin{aligned}
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \phi^0 - \frac{(m_1+m_2)}{2} \phi - \left(\frac{d}{dr} + \frac{2}{r} \right) \phi_e - \frac{1}{r} \phi_l &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \phi - \frac{(m_1+m_2)}{2} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \chi^0 - \frac{(m_1-m_2)}{2} \chi - \left(\frac{d}{dr} + \frac{2}{r} \right) \chi_e - \frac{1}{r} \chi_l &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \chi - \frac{(m_1-m_2)}{2} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_e - \frac{(m_1+m_2)}{2} \chi_e^0 + \frac{d}{dr} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_l - \frac{(m_1+m_2)}{2} \chi_l^0 - \frac{j(j+1)}{r} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_m - \frac{(m_1+m_2)}{2} \chi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_e^0 - \frac{(m_1+m_2)}{2} \chi_e + \frac{1}{r} \phi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_l^0 - \frac{(m_1+m_2)}{2} \chi_l - \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_m^0 - \frac{(m_1+m_2)}{2} \chi_m + \frac{j(j+1)}{r} \phi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_l^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_e - \frac{(m_1-m_2)}{2} \phi_e^0 + \frac{d}{dr} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_l - \frac{(m_1-m_2)}{2} \phi_l^0 - \frac{j(j+1)}{r} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_m - \frac{(m_1-m_2)}{2} \phi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_e^0 - \frac{(m_1-m_2)}{2} \phi_e + \frac{1}{r} \chi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_l^0 - \frac{(m_1-m_2)}{2} \phi_l - \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_m^0 - \frac{(m_1-m_2)}{2} \phi_m + \frac{j(j+1)}{r} \chi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_l^0 &= 0
\end{aligned} \tag{20}$$

At this point, we will concentrate on the properties of the scalar and vector components as well as the new components to which they have been expanded. As pointed out earlier, the scalar components $\phi, \phi^0, \chi, \chi^0$ describe the $S = 0$ states, while the rest, the vector components, describe the states with $S = 1$. In the latter case, for $\vec{\chi}$, there are three states characterised by the spectroscopic signatures $^3(j-1)_j, ^3(j+1)_j$ and 3j_j (in the atomic notation $^{2S+1}L_j$) described by the wave-functions $\chi_{l=j-1}, \chi_{l=j+1}$ and χ_l satisfying the following relations [36, 37]

$$\begin{aligned}
\chi_e &= \sqrt{\frac{j}{2j+1}} \chi_{l=j-1} + \sqrt{\frac{j+1}{2j+1}} \chi_{l=j+1} \\
\chi_l &= \sqrt{j(j+1)} \left(-\sqrt{\frac{j+1}{2j+1}} \chi_{l=j-1} + \sqrt{\frac{j}{2j+1}} \chi_{l=j+1} \right) \\
\chi_m &= \sqrt{j(j+1)} \chi_{l=j}
\end{aligned} \tag{21}$$

Certainly, this can be generalised for the rest of the $S = 1$ components: $\vec{\chi}^0, \vec{\phi}$ and $\vec{\phi}^0$. The “magnetic” component is chosen to have $l = j$ while the “electric” and the “longitudinal” components have $l = j \pm 1$ mixed if $j > 0$ (and $l = 1$, if $j = 0$). If $j = 0$, then there should be only one χ component and we note this is satisfied since $\chi_l = \chi_m = 0$ and $\chi_{l=j-1} = \chi_{l=j} = 0, \chi_{l=j+1} = \chi_e$. There are only two states with $j = 0$, 1S_0 ($j = 0, l = 0, S = 0$) and 3P_0 ($j = 0, l = 1, S = 1$).

The expressions (8) and (9) indicate that, for $S = 0$, the “large-large” (ψ_{++}) and “small-small” (ψ_{--}) components are contained in the $\vec{\chi}, \vec{\chi}^0$ and the “large-small” (ψ_{+-}) and “small-large” (ψ_{-+}) components appear in the $\vec{\phi}, \vec{\phi}^0$. In addition, all “electric” and “longitudinal” components of the $\vec{\chi}, \vec{\chi}^0, \vec{\phi}, \vec{\phi}^0$ correspond to states with magnetic quantum number $m_s = \pm 1$ while the “magnetic” components describe states with $m_s = 0$. On the other hand, the states with $S = 0$ have the spectroscopic notation 1j_j and the “large-large” (ψ_{++}) and “small-small” (ψ_{--}) components are contained in the ϕ, ϕ^0 and the “large-small” (ψ_{+-}) and “small-large” (ψ_{-+}) components appear in the χ, χ^0 . All $\phi, \phi^0, \chi, \chi^0$ correspond to states with $m_s = 0$.

The parity of the system equals $P = \eta(-1)^l$, where $\eta = 1$ or -1 for fermion-fermion or fermion-antifermion system, respectively. The sixteen components have the following parity:

- (i). $\phi, \phi^0 : P = \eta(-1)^j$ because they both describe the $S = 0$ states, therefore $l = j$,
- (ii). $\phi_e, \phi_e^0, \phi_l, \phi_l^0 : P = -\eta(-1)^{j+1}$ (or $-\eta(-1)^{j-1}$ which gives the same result) $= \eta(-1)^j$. According to (21) the “electric” and “longitudinal” components are combinations of states with $l = j + 1$ and $l = j - 1$ and this justifies the exponent $j + 1$ (or $j - 1$). The $(-)$ sign in front of η is due to the fact that all these functions are combinations of “small” and “large” components,
- (iii). $\chi_m, \chi_m^0 : P = \eta(-1)^j$ ((21) implies that the “magnetic” component corresponds to $l = j$ states),
- (iv). $\chi, \chi^0 : P = -\eta(-1)^j$. Both χ, χ^0 describe $S = 0$ states therefore $l = j$, however, in contrast to the parity of the ϕ, ϕ^0 , there is a minus sign which is attributed to the fact that χ, χ^0 are combinations of “small” and “large” components,
- (v). $\chi_e, \chi_e^0, \chi_l, \chi_l^0 : P = -\eta(-1)^j$, (as in (ii)),
- (vi). $\phi_m, \phi_m^0 : P = -\eta(-1)^j$, (as in (iii)).

Again, the combination of “small” and “large” components accounts for the minus sign in (v), (vi). To summarise, the components $\phi, \phi^0, \phi_e, \phi_e^0, \phi_l, \phi_l^0, \chi_m, \chi_m^0$ have parity $P = \eta(-1)^j$ while the components $\chi, \chi^0, \chi_e, \chi_e^0, \chi_l, \chi_l^0, \phi_m, \phi_m^0$ have parity $P = -\eta(-1)^j$. We call the former case, Pseudoscalar Particle Trajectory (PPT) while the latter is called Vector Particle Trajectory (VPT) [36, 37]. The “large-large” and “small-small” components in the PPT are contained in the $\phi, \phi^0, \chi_m, \chi_m^0$ and have spectroscopic signature 1j_j or 3j_j . In the VPT, the “large-large” and “small-small” components are contained in $\chi_e, \chi_e^0, \chi_l, \chi_l^0$ and have signature $^3(j-1)_j$ or $^3(j+1)_j$.

Returning to Eqs.(20) and by making use of the previous discussion, that set of equations can be split into two sets of equations according to whether they belong to the PPT or VPT regime. To be more specific, if the two fermions do not possess the same mass, the following two sets of equations are obtained from Eqs.(20) [10, 36, 37]

(i). PPT, 1j_j or $^3j_j, P = \eta(-1)^j$.

$$\begin{aligned}
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \phi^0 - \frac{(m_1+m_2)}{2} \phi - \left(\frac{d}{dr} + \frac{2}{r} \right) \phi_e - \frac{1}{r} \phi_l &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \phi - \frac{(m_1+m_2)}{2} \phi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_m - \frac{(m_1+m_2)}{2} \chi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_m^0 - \frac{(m_1+m_2)}{2} \chi_m + \frac{j(j+1)}{r} \phi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_l^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_e - \frac{(m_1-m_2)}{2} \phi_e^0 + \frac{d}{dr} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_l - \frac{(m_1-m_2)}{2} \phi_l^0 - \frac{j(j+1)}{r} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_m - \frac{(m_1-m_2)}{2} \phi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_e^0 - \frac{(m_1-m_2)}{2} \phi_e + \frac{1}{r} \chi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_l^0 - \frac{(m_1-m_2)}{2} \phi_l - \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_m^0 &= 0
\end{aligned} \tag{22}$$

(ii). VPT, ${}^3(j \pm 1)_j, P = -\eta(-1)^j$.

$$\begin{aligned}
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \chi^0 - \frac{(m_1 - m_2)}{2} \chi - \left(\frac{d}{dr} + \frac{2}{r} \right) \chi_e - \frac{1}{r} \chi_l &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \chi - \frac{(m_1 - m_2)}{2} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_e - \frac{(m_1 + m_2)}{2} \chi_e^0 + \frac{d}{dr} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_l - \frac{(m_1 + m_2)}{2} \chi_l^0 - \frac{j(j+1)}{r} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_e^0 - \frac{(m_1 + m_2)}{2} \chi_e + \frac{1}{r} \phi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_l^0 - \frac{(m_1 + m_2)}{2} \chi_l - \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_m^0 - \frac{(m_1 - m_2)}{2} \phi_m + \frac{j(j+1)}{r} \chi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_l^0 &= 0
\end{aligned} \tag{23}$$

On the other hand, in the case of equal masses $m_1 = m_2 = m$, the set PPT splits into two subsets, one with spectroscopic signature 1j_j and a second with 3j_j while the VPT remains unchanged

(i). PPT, ${}^1j_j, P = \eta(-1)^j$.

$$\begin{aligned}
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \phi^0 - m\phi - \left(\frac{d}{dr} + \frac{2}{r} \right) \phi_e - \frac{1}{r} \phi_l &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \phi - m\phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_e + \frac{d}{dr} \phi^0 &= 0 \\
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \phi_l - \frac{j(j+1)}{r} \phi^0 &= 0
\end{aligned} \tag{24}$$

(ii). PPT, ${}^3j_j, P = \eta(-1)^j$.

$$\begin{aligned}
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_m - m\chi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_m^0 - m\chi_m + \frac{j(j+1)}{r} \phi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_l^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_e^0 + \frac{1}{r} \chi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_l^0 - \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_m^0 &= 0
\end{aligned} \tag{25}$$

(iii). VPT, ${}^3(j \pm 1)_j, P = -\eta(-1)^j$.

$$\begin{aligned}
\frac{1}{2} \left[E - S(r) + P(r) - V(r) \right] \chi^0 - \left(\frac{d}{dr} + \frac{2}{r} \right) \chi_e - \frac{1}{r} \chi_l &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_e - m \chi_e^0 + \frac{d}{dr} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) - P(r) - V(r) \right] \chi_l - m \chi_l^0 - \frac{j(j+1)}{r} \chi^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_e^0 - m \chi_e + \frac{1}{r} \phi_m^0 &= 0 \\
\frac{1}{2} \left[E + S(r) + P(r) - V(r) \right] \chi_l^0 - m \chi_l - \left(\frac{d}{dr} + \frac{1}{r} \right) \phi_m^0 &= 0 \\
\frac{1}{2} \left[E - S(r) - P(r) - V(r) \right] \phi_m^0 + \frac{j(j+1)}{r} \chi_e^0 + \left(\frac{d}{dr} + \frac{1}{r} \right) \chi_l^0 &= 0
\end{aligned} \tag{26}$$

The remaining two components χ, ϕ_m vanish in the event of the particles having the same mass m .

3 A Potential for quarkonium

In the Introduction, we discussed the lack of covariance characterising the Breit equation. The wavefunction of the system of the two fermions depends on the position of the two particles (that is, two vectors, each for the two particles, are necessary) while it depends only on one time variable. This fact does not allow a Lorentz-invariant formulation. However, if a very strong, short-range potential is considered to govern the interaction of the two particles, we can assume the two particles are so close to one another that just one position vector is enough to describe the motion and the quantum mechanics of the system. In this case, the number of the position vector variables equals that of the time variables and we can assume that the equation becomes approximately covariant. The question rising is whether there exist such interactions between particles and if there are systems of fermions to which the Breit equation can be applied and subsequently be tested. Talking about small distances, we are led to the plausible question of what happens at distances up to 1 fermi which is a feature of the strong interactions between quarks. It is known that there are systems, quarkonia, which are bound states of quarks and antiquarks. Actually, experimental data about many of these states exist [41], therefore, one could maintain that they constitute a laboratory where the validity of any theory aiming at describing them can be tested.

In the non-relativistic limit, the quark-antiquark interaction is dominated by vector and scalar potential [10, 33]. The former, which is of the Coulomb type constitutes the interaction at small distances and corresponds to one gluon exchange. In that case, quarkonium can be viewed as the “hydrogen atom” of strong interaction Physics. By contrast, at large distances, confinement dominates and the interaction is proportional to the interparticle distance. The confining part of the potential has scalar structure [10, 33] and thereby the interquark potential consists of a Coulomb part which is of vector type and a linear part which is of scalar type

$$V_{\text{int}}(r) = V_{\text{Vector}}(r) + V_{\text{Scalar}}(r) = -\frac{4}{3} \frac{\alpha_s(Q^2)}{r} [(\gamma_0^{(1)} \gamma_\mu^{(1)}) \otimes (\gamma_0^{(2)} \gamma_\mu^{(2)})] + \kappa r (\gamma_0^{(1)} \otimes \gamma_0^{(2)}) \quad (27)$$

where α_s is the coupling constant of the strong interaction, Q^2 is the relevant momentum transfer and κ is the string constant [39].

Certainly, the non-relativistic part of the potential is not sufficient if we wish to test the Breit equation. The contribution of relativistic corrections are required and we must take them into account. At this point, we recall that the original Breit equation contained terms called Breit interaction that account for relativistic corrections. The inclusion of the Breit interaction in the original equation does not lead to correct results ([10]-[15]) for certain quantum-mechanical systems. Unless this term is treated as a small perurbation, the results differ from the well-established data.

By studying the scattering amplitude for a particle-antiparticle collision [10, 40], we can derive an effective potential in the Pauli approximation of the Breit equation. That potential will account for the interaction and it will include both relativistic and non-relativistic contributions. We need bear in mind that the Breit interaction is the second-order term in the (v/c) expansion of a Coulomb interaction. In the Pauli approximation, the vector part of the quark-antiquark potential contributes the following term to the

interaction

$$\begin{aligned}
U_{\text{Vector}}(\vec{p}, \vec{r}) &= \left\{ -\frac{4}{3} \frac{\alpha_s}{r} \right\} + V_1 + V_2 + V_3 \\
V_1 &= \left\{ -\frac{\vec{p}^4}{8m_1^3} - \frac{\vec{p}^4}{8m_2^3} + \frac{2}{3} \pi \alpha_s \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(\vec{r}) \right\} - \frac{2\alpha_s}{3m_1 m_2 r} \left[\vec{p}^2 + \frac{(\vec{r} \cdot \vec{p})^2}{r^2} \right] \\
V_2 &= \left\{ \frac{2\alpha_s}{3r^3} \left[\frac{1}{m_1^2} \vec{L} \cdot \vec{S}_1 + \frac{1}{m_2^2} \vec{L} \cdot \vec{S}_2 \right] \right\} + \frac{4\alpha_s}{3m_1 m_2 r^3} \vec{L} \cdot \vec{S} \\
V_3 &= \frac{2\alpha_s}{m_1 m_2 r^3} \left[-\frac{1}{3} \vec{S}^2 + \frac{(\vec{r} \cdot \vec{S})^2}{r^2} \right] + \frac{16\pi\alpha_s}{9m_1 m_2} \left(\vec{S}^2 - \frac{3}{2} \right) \delta(\vec{r})
\end{aligned} \tag{28}$$

where the $\{\dots\}$ terms correspond to the static part of the interaction. The rest of the terms are responsible for retardation effects and they can be associated to the Breit interaction. We emphasise that this derivation is in analogy with the procedure followed towards obtaining the retarded part of the potential in a particle-antiparticle electromagnetic interaction (scattering). We recall, that in that case, the contribution of the D_{ik} ($i, k = 1, 2, 3$) part of the photon propagator in the Coulomb gauge was taken into account [40].

On the other hand, the scalar confining potential contributes only to the static part of the interaction. Unlike the vector part of the interaction, the scalar term of the potential cannot lead to the emergence of retarded effects since there is not an equivalent of the photon propagator that would result in such a contribution. The scattering amplitude leads to the following interaction operator [10, 33]

$$\begin{aligned}
U_{\text{Scalar}}(\vec{r}) &= \{\kappa r\} + V_4 \\
V_4 &= \left\{ -\frac{1}{2m_1^2 m_2^2 r} \vec{L} \cdot (m_1^2 \vec{S}_2 + m_2^2 \vec{S}_1) \right\}
\end{aligned} \tag{29}$$

where $\{\dots\}$ merely imply that the interaction is characterised by static behaviour. For particles with the same mass, V_2, V_4 correspond to the spin-orbit interaction. Although both potentials, (V_{Vector}) and (V_{Scalar}) are attractive, they give the opposite sign of spin-orbit interaction, a feature which is going to result in a quite interesting effect regarding the ordering of the states (Section 5.1). In addition, we should stress the absence of a spin-spin interaction in the confining potential meaning that tensor forces are contained entirely in the Coulomb-like vector potential. The absence of spin-spin and tensor terms in the scalar interaction as well as the different form of spin-orbit terms between the linear and Coulomb-like potentials constitute the most striking difference between the two types of interaction.

So far, the form of the potential describing strong interactions has been discussed to some extent but nothing has been mentioned about the efficiency of such a potential. The charmonium and bottomium spectra are well-known [41], therefore the accuracy of the potential besides the adequacy of the Breit equation in describing strong interactions can be tested. Next, we will follow a procedure towards calculating the energy levels of various bound states of quark-antiquark systems, however, we have to obtain first the equations. In the following section, we will derive the radial equations for those systems as they are produced by means of the Eqs. [(24), (25), (26)].

4 Application of the Breit equation to quarkonium

The quark-antiquark bound states can be classified in three categories according to their spectroscopic signature [10]:

(i). States with $j = l, S = 0$. After making the substitutions $V(r) = -\frac{4}{3}\frac{\alpha_s}{r}$, $S(r) = -\kappa r$ and eliminating all components but ϕ^0 , Eqs.(24) lead to

$$\begin{aligned} \frac{d^2\phi^0(r)}{dr^2} + \frac{d\phi^0(r)}{dr} \left(\frac{2}{r} - \frac{\kappa - \frac{4}{3}\frac{\alpha_s}{r^2}}{E + \kappa r + \frac{4}{3}\frac{\alpha_s}{r}} \right) + \left\{ \frac{1}{4} \left[\left(E + \frac{4}{3}\frac{\alpha_s}{r} \right)^2 - \kappa^2 r^2 \right] - \right. \\ \left. - \frac{j(j+1)}{r^2} - m^2 \frac{E + \kappa r + \frac{4}{3}\frac{\alpha_s}{r}}{E - \kappa r + \frac{4}{3}\frac{\alpha_s}{r}} \right\} \phi^0(r) = 0 \end{aligned} \quad (30)$$

(ii). States with $j = l, S = 1$.

Similarly, Eqs.(25) give

$$\begin{aligned} \frac{d^2\chi_m^0(r)}{dr^2} + \frac{d\chi_m^0(r)}{dr} \left(\frac{2}{r} - \frac{\kappa - \frac{4}{3}\frac{\alpha_s}{r^2}}{E + \kappa r + \frac{4}{3}\frac{\alpha_s}{r}} \right) + \left\{ \frac{1}{4} \left[\left(E + \frac{4}{3}\frac{\alpha_s}{r} \right)^2 - \kappa^2 r^2 \right] - \frac{j(j+1)}{r^2} - \right. \\ \left. - m^2 \frac{E + \kappa r + \frac{4}{3}\frac{\alpha_s}{r}}{E - \kappa r + \frac{4}{3}\frac{\alpha_s}{r}} - \frac{1}{r} \frac{\kappa - \frac{4}{3}\frac{\alpha_s}{r^2}}{E + \kappa r + \frac{4}{3}\frac{\alpha_s}{r}} \right\} \chi_m^0(r) = 0 \end{aligned} \quad (31)$$

(iii). States with $j = l \pm 1, S = 1$.

Finally, Eqs.(26) can be reduced to the following two coupled differential equations

$$\begin{aligned} \frac{d^2\chi_e^0(r)}{dr^2} + \frac{d\chi_e^0(r)}{dr} \left[\frac{2}{r} + \frac{-\frac{4}{3}\frac{\alpha_s}{r^2} - \kappa}{E + \frac{4}{3}\frac{\alpha_s}{r} - \kappa r} + \frac{-2\kappa \left(E + \frac{8}{3}\frac{\alpha_s}{r} \right)}{\left(E + \frac{4}{3}\frac{\alpha_s}{r} \right)^2 - (\kappa r)^2} \right] + \\ + \chi_e^0(r) \left\{ \frac{\left(E + \frac{4}{3}\frac{\alpha_s}{r} + \kappa r \right) \left[\frac{1}{4} \left(E + \frac{4}{3}\frac{\alpha_s}{r} - \kappa r \right)^2 - m^2 \right]}{E + \frac{4}{3}\frac{\alpha_s}{r} - \kappa r} - \frac{j(j+1)}{r^2} + \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{-4\kappa \left(E + \frac{8\alpha_s}{3r}\right)}{\left(E + \frac{4\alpha_s}{3r}\right)^2 - (\kappa r)^2} \frac{1}{r} - \frac{2}{r^2} + \frac{\kappa^2 + \frac{16\alpha_s^2}{9r^4} + \frac{8\alpha_s\kappa}{3r^2} + \frac{8\alpha_s E}{3r^3}}{E + \frac{4\alpha_s}{3r} + \kappa r} \frac{1}{E + \frac{4\alpha_s}{3r} - \kappa r} \Bigg\} = \\
& = \chi_l^0(r) \left[\frac{1}{r^2} - \frac{-2\kappa \left(E + \frac{8\alpha_s}{3r}\right)}{\left(E + \frac{4\alpha_s}{3r} + \kappa r\right)^2} \frac{1}{r} + \frac{\left(E + \frac{4\alpha_s}{3r} - \kappa r\right)}{\left(E + \frac{4\alpha_s}{3r} + \kappa r\right)} \frac{1}{r^2} \right] \quad (32)
\end{aligned}$$

$$\begin{aligned}
& \frac{d^2\chi_l^0(r)}{dr^2} + \frac{d\chi_l^0(r)}{dr} \left[\frac{2}{r} + \frac{-\frac{4\alpha_s}{3r^2} - \kappa}{E + \frac{4\alpha_s}{3r} + \kappa r} \right] + \chi_l^0(r) \left\{ \frac{E + \frac{4\alpha_s}{3r} + \kappa r}{E + \frac{4\alpha_s}{3r} - \kappa r} \times \right. \\
& \times \left[\frac{1}{4} \left(E + \frac{4\alpha_s}{3r} - \kappa r\right)^2 - m^2 \right] - \frac{j(j+1)}{r^2} - \frac{1}{r} \frac{-\frac{4\alpha_s}{3r^2} + \kappa}{E + \frac{4\alpha_s}{3r} + \kappa r} \Bigg\} = \\
& = \chi_e^0(r) \left[\frac{2j(j+1)}{r^2} + \frac{j(j+1)}{r} \left(\frac{-\frac{4\alpha_s}{3r^2} - \kappa}{E + \frac{4\alpha_s}{3r} - \kappa r} + \frac{-\frac{4\alpha_s}{3r^2} + \kappa}{E + \frac{4\alpha_s}{3r} + \kappa r} \right) \right] \quad (33)
\end{aligned}$$

The behaviour of the component wavefunctions in the above equations near the origin can be deduced easily if we notice that at very small interparticle distances, the dominant part of the potential has a vector, Coulomb-like form. Thus, the wavefunctions for small r look like

$$\begin{aligned}
& \text{state } ^1l_l \quad , \quad r\phi^0 \sim r^{\gamma+1} \quad , \quad \gamma = -1 + \sqrt{1 + j(j+1) - \frac{4}{9}\alpha_s^2} \\
& \text{state } ^3l_l \quad , \quad r\chi_m^0 \sim r^{\gamma+1} \quad , \quad \gamma = -1 + \sqrt{j(j+1) - \frac{4}{9}\alpha_s^2} \\
& \text{state } ^3l_{l\pm 1} \quad , \quad \left\{ \begin{array}{l} r\chi_e^0 \sim r^{\gamma+1} \\ r\chi_l^0 \sim r^{\gamma+1} \end{array} \right\} \quad , \quad \gamma = \left\{ \begin{array}{l} \sqrt{j(j+1) + 1 - \frac{4}{9}\alpha_s^2} \\ -1 + \sqrt{j(j+1) - \frac{4}{9}\alpha_s^2} \end{array} \right. \quad (34)
\end{aligned}$$

On the other hand, due to the inclusion of the confining potential, the wavefunctions have a different behaviour at large distances. They look like $\sim e^{-\frac{1}{4}\kappa r^2}$. The behaviour of the wavefunctions near the origin and at large distances will be used in the next sections in order to solve the differential equations describing the dynamics of the quantum mechanical systems.

5 Solution of the equations

As pointed out in the previous sections, the energy levels of the various states are derived from the solution of the above radial equations, however, those values do not represent the complete energy since the potential used is merely the static potential. The contribution of the retarded part of the interaction should be included to obtain the relativistic corrections

to the energy. The static part consists of the $\{\dots\}$ terms in the expressions [(28), (29)] while the rest are the retarded terms. At this point, we should recall that we do not expect satisfactory results unless the Breit (retarded) terms are treated by first order perturbation theory for particular states. The application to QED problems ([10]-[15]) demonstrated this assertion is correct. In that case, the total energy of the system is given by the expression

$$E_{\text{total}} = E_{\text{static}} + \langle \text{state} | V_{\text{retarded}} | \text{state} \rangle \quad (35)$$

where $|\text{state}\rangle$ stands for the stationary unperturbed states for the Coulomb potential. The same procedure can be followed when the potential has the form (27), however, unlike QED problems, the eigenstates are not known and they should be calculated before we proceed. Unfortunately, an exact solution for such a potential cannot be obtained and an approximate method should be tried. One way is by treating the Coulomb term as a small perturbation [42]. Another method, which will be employed in this paper, is to use the three-dimensional isotropic harmonic oscillator (TDIHO) eigenstates [10, 43]. The potential of the harmonic oscillator provides confinement of quarks and the important feature of this potential is that the wavefunctions and the matrix elements can be calculated easily and they can possess an explicit form. The parameters of the wavefunctions will be adjusted so as to fit the numerically calculated eigenfunctions for the potential (27).

The interaction potential of TDIHO equals

$$V_{\text{TDIHO}} = \frac{1}{2} \mu \omega^2 r^2 \quad (36)$$

where μ and ω are fitting parameters. The energy levels of the system are given by the expression

$$E(n_r, l) = \left(2n_r + l + \frac{3}{2} \right) \omega, \quad n_r, l = 0, 1, 2, \dots \quad (37)$$

and the normalised radial parts of the wavefunctions which are going to be used are the

following [43]

$$\begin{aligned}
R_{1S} &= 2 \underbrace{\left(\frac{\lambda_{1S}^3}{\pi} \right)^{1/4} \exp \left(-\frac{\lambda_{1S}}{2} r^2 \right)}_{B_{1S}(r)} , & R_{1P} &= \sqrt{\frac{8\lambda_{1P}}{3}} r B_{1P}(r) \\
R_{2S} &= \sqrt{6} \left(1 - \frac{2}{3} \lambda_{2S} r^2 \right) B_{2S}(r) , & R_{2P} &= \sqrt{\frac{4\lambda_{2P}}{15}} r (5 - 2\lambda_{2P} r^2) B_{2P}(r) \\
R_{3S} &= \frac{1}{\sqrt{30}} (15 - 20\lambda_{3S} r^2 + 4\lambda_{3S}^2 r^4) B_{3S}(r) , & R_{4S} &= \frac{1}{\sqrt{1260}} (105 - 210\lambda_{4S} r^2 + \\
& & & + 84\lambda_{4S}^2 r^4 - 8\lambda_{4S}^3 r^6) B_{4S}(r) \\
R_{5S} &= \frac{1}{\sqrt{90720}} (945 - 2520\lambda_{5S} r^2 + 1512\lambda_{5S}^2 r^4 - & R_{6S} &= \frac{1}{\sqrt{9979200}} (10395 - 34650\lambda_{6S} r^2 + \\
& - 288\lambda_{5S}^3 r^6 + 16\lambda_{5S}^4 r^8) B_{5S}(r) , & & + 27720\lambda_{6S}^2 r^4 - 7920\lambda_{6S}^3 r^6 + \\
& & & + 880\lambda_{6S}^4 r^8 - 32\lambda_{6S}^5 r^{10}) B_{6S}(r)
\end{aligned}$$

where $B_{\text{index}} \equiv B(\lambda_{1S} \rightarrow \lambda_{\text{index}})$. $\lambda_{1S} \equiv \lambda = \mu\omega$ while the rest of the λ s are functions of λ_{1S} and their values will be determined from experimental data. This is done because the harmonic oscillator is an approximation of the “funnel” potential [16, 28] therefore the values of the various parameters need to be adjusted to be in agreement with the experimental values of the corresponding quantities. It is necessary to emphasise that the λ s have a mass dependence implying that the relation among them which will emerge is not the same for all quarkonium systems. The parameters μ, ω , in turn, can be estimated by comparing the values of the quantities in which they enter with those derived from experimental data. The leptonic widths, the mass differences and the various energy levels of some certain states are quantities that can help us to estimate not only those parameters but also other parameters such as the coupling constant α_s , the string constant κ and the quark masses. To achieve this goal, we will take advantage of the knowledge of the energy levels of the charmonium and bottomium systems [41].

5.1 The Bottomium system

We aim to estimate the energies of the twelve states of the bottomium system which are spin triplet states, i.e. $S = 1$ (*ortho-bottomium*). The $1^{--}, 0^{++}, 2^{++}$ states are vector mesons and the static part of the interaction between the constituent particles leads to the description of their dynamics by Eqs. [(32), (33)]. On the other hand, the rest of the states, denoted with 1^{++} , are pseudoscalar mesons and they are described by Eq.(31). Finally, six more pseudoscalar meson states (Table 2) (with $J^{PC} = 0^{-+}, 1^{+-}$) will be considered and Eq.(30) describes those states. Unlike QED, the explicit form of the potential of QCD and

the coupling strength are not known therefore a method towards estimating them should be formulated. The procedure becomes more difficult if we remember that the solution of the above equations is not enough to get the energy spectrum since the values calculated account only for a part of the whole energy. The rest of the contribution results from relativistic effects due to the non-instantaneous interaction which should not be ignored. The combination leads to the problem resisting an analytic treatment and only a numerical method seems capable of solving it.

Since the coupling constant α_s of the system as well as the mass of the bottom quark m_b are not known, the use of some of the states to determine these values is inevitable. In addition, there are some more parameters which need to be estimated: the string constant κ and the λ s. Before we embark on calculating the various quantities, we will first attempt to express the energies of the states in terms of the unknown parameters.

The energy of the system consists of two parts, one coming from the non-perturbative solution of the Eqs. [(30)-(33)] and another derived from the perturbative treatment of the non-retarded parts of V_1, V_2, V_3 in (28) by replacing m_1, m_2 with $m_b \equiv m$. The former will be called E_{static} while the latter which is non-instantaneous, in nature, will be called $E_{\text{non-static}}$. The combined result in the total energy of the system E_{system} is

$$\begin{aligned}
E_{\text{system}} = E_{\text{static}} + \underbrace{\langle V_1 + V_2 + V_3 \rangle}_{E_{\text{non-static}}} = E_{\text{static}} + \left\langle -\frac{2\alpha_s}{3m^2} \left[-\frac{2}{r} \frac{d^2}{dr^2} - \frac{2}{r^2} \frac{d}{dr} + \frac{1}{r^3} \vec{L}^2 \right] + \right. \\
\left. + \frac{4\alpha_s}{3m^2 r^3} \vec{L} \cdot \vec{S} + \frac{2\alpha_s}{m^2 r^3} \underbrace{\left[-\frac{1}{3} \vec{S}^2 + \frac{(\vec{r} \cdot \vec{S})^2}{r^2} \right]}_{S_{12}} + \frac{16\pi\alpha_s}{9m^2} \left(\vec{S}^2 - \frac{3}{2} \right) \delta(\vec{r}) \right\rangle \quad (38)
\end{aligned}$$

The spin-orbit perturbative correction and S_{12} vanish for S -states while the δ functions give a non-zero contribution. By contrast, the P -states are characterised by the opposite behaviour ($\vec{L} \cdot \vec{S}, S_{12} \neq 0, \langle \delta(\vec{r}) \rangle = 0$). In Table 1, the energies of various states are summarised and we notice that the knowledge of the λ s is necessary to obtain the values of the corrections due to the Breit terms. The decay widths of the states will be used, next, to evaluate the relation of the λ s and other parameters to some measured quantities. The ‘‘Corrected Van Royen-Weisskopf’’ formula [18, 26]

$$\Gamma(V \rightarrow e^+ e^-) = \frac{16\pi\alpha^2 Q_b^2}{m_V^2} |\psi(0)|^2 \left(1 - \frac{16}{3\pi} \alpha_s \right), \quad \alpha = \frac{1}{137}, \quad Q_b = \frac{1}{3} \quad (39)$$

relates the leptonic width, i.e. e^+e^- decay of the neutral V vector mesons, and the wave-function of bottomium at the origin. m_V stands for the mass of the meson. Although the coupling constant has not been estimated yet, the fact that we are dealing with strong interactions suggests that the radiative corrections are so large that only ratios such as

$$r(V'/V) \equiv \frac{\Gamma(V' \rightarrow e^+e^- \text{ or } \mu^+\mu^-)}{\Gamma(V \rightarrow e^+e^- \text{ or } \mu^+\mu^-)} = \frac{m_V^2}{(m'_V)^2} \frac{|\psi'(0)|^2}{|\psi(0)|^2} \quad (40)$$

can be calculated reliably because the corrections are suppressed and do not appear in the above expressions. In (40), V' is another vector state of the quarkonium system having mass m'_V . All 1^{--} vector states can decay into an electron-positron (or muon-antimuon) pair and the five ratios of these decays with respect the “reference decay” of the $\Upsilon(1S)$ state are given from experiment [41]

$$r(2S/1S)_\mu \equiv \frac{E_{\Upsilon(1S)}^2}{E_{\Upsilon(2S)}^2} \frac{|\psi_{\Upsilon(2S)}(0)|^2}{|\psi_{\Upsilon(1S)}(0)|^2} = 0.4 \pm 0.1 \quad (41)$$

$$r(3S/1S)_\mu \equiv \frac{E_{\Upsilon(1S)}^2}{E_{\Upsilon(3S)}^2} \frac{|\psi_{\Upsilon(3S)}(0)|^2}{|\psi_{\Upsilon(1S)}(0)|^2} = 0.37 \pm 0.06 \quad (42)$$

$$r(4S/1S) \equiv \frac{E_{\Upsilon(1S)}^2}{E_{\Upsilon(4S)}^2} \frac{|\psi_{\Upsilon(4S)}(0)|^2}{|\psi_{\Upsilon(1S)}(0)|^2} = 0.18 \pm 0.04 \quad (43)$$

$$r(5S/1S) \equiv \frac{E_{\Upsilon(1S)}^2}{E_{\Upsilon(5S)}^2} \frac{|\psi_{\Upsilon(5S)}(0)|^2}{|\psi_{\Upsilon(1S)}(0)|^2} = 0.23 \pm 0.05 \quad (44)$$

$$r(6S/1S) \equiv \frac{E_{\Upsilon(1S)}^2}{E_{\Upsilon(6S)}^2} \frac{|\psi_{\Upsilon(6S)}(0)|^2}{|\psi_{\Upsilon(1S)}(0)|^2} = 0.10 \pm 0.02 \quad (45)$$

(the index μ in (41) and (42) indicates that the vector mesons involved decay into muon-antimuon pairs). Each one of $r(nS/1S)$ ($n = 2, \dots, 6$) is a function of the energy of the system and two λ s, i.e., λ_{1S} and λ_{nS} .

The relations [(41)-(45)] can be used to establish the relation among the λ s, energies and m . (38) is another useful expression which will be employed in order to calculate the energies of the various states as well as the values of α_s, κ, m . The part of the energy due to the instantaneous interaction should be combined with the non-retarded corrections so that the energy obtained is in agreement with experiment. This is called “fitting procedure”

and it will help us to determine the four unknown parameters $\alpha_s, \kappa, m, \lambda_{1S}$. The rest of the λ s, associated with the S -states, are related to them through the decay widths $\Gamma(V \rightarrow e^+e^-)$ whose values are well established. There are two more λ s, namely, λ_{1P} and λ_{2P} which cannot be calculated accurately by means of the transitions decays, as pointed out in the previous paragraph. The total number of the unknown parameters, i.e. six, requires six energy values to be set as input values. We choose the energies of the $\Upsilon(1S), \Upsilon(2S), \Upsilon(3S), \Upsilon(4S), \chi_{b1}(1P), \chi_{b1}(2P)$ states as the input values. Although P -states with the same principal quantum number can be used, we avoid it because they correspond to energies very close to one another which may lead to wrong results. The harmonic oscillator wavefunctions constitute an approximation to the real wavefunctions of the funnel potential and in the stage of optimising them, the method may be too sensitive to small energy differences. Thus, we choose the following values established by experiment [41]

$$\begin{aligned}
E[\Upsilon(1S)] &= 9460.37 \pm 0.21 \text{ MeV} \\
E[\Upsilon(2S)] &= 10023.30 \pm 0.31 \text{ MeV} \\
E[\Upsilon(3S)] &= 10355.3 \pm 0.5 \text{ MeV} \\
E[\Upsilon(4S)] &= 10580.0 \pm 3.5 \text{ MeV} \\
E[\chi_{b1}(1P)] &= 9891.9 \pm 0.7 \text{ MeV} \\
E[\chi_{b1}(2P)] &= 10255.2 \pm 0.5 \text{ MeV}
\end{aligned} \tag{46}$$

The energies of the Υ states will help us to determine the relations between the λ_{nS} ($n = 2, 3, 4$) and λ_{1S} through the ratios [(41)-(43)].

The mass of the bottom quark is regarded as a parameter which will be evaluated, however, its value will be constrained in the range $4.1 \text{ GeV} \leq m_b \equiv m \leq 5 \text{ GeV}$ (running mass) [41]. The energies due to the static interaction can be expressed easily in terms of the total energy and the parameters. From (38), $E_{\text{static}} = E_{\text{total}} - E_{\text{perturbation}}$ which means that the numerical solution of Eqs. [(31)-(33)] can lead to the determination of the parameters, provided that the relations connecting them dictated by the corresponding quantities are satisfied. Thus, the main problem reduces to the solution of the differential equations. Certainly, it is important to observe that Eqs. [(30)-(33)] exhibit a singularity at

$$r_0 = \frac{1}{2} \frac{E}{\kappa} + \sqrt{\frac{1}{4} \left(\frac{E}{\kappa} \right)^2 + \frac{4}{3} \frac{\alpha_s}{\kappa}} \tag{47}$$

This is a feature which is not present in the initial Breit equation but it emerges after

the reduction of it to radial equations. Although the equations are singular at $r = 0$ as well, due to the centrifugal term $\frac{j(j+1)}{r^2}$ (j is the total angular momentum of the system), the new singularity implies the appearance of a turning point that is energy dependent [10, 13, 44]. The question rising is whether the emergence of this singularity causes any problem to the solution of the differential equations. Due to the fact that we consider only a short-range potential and the wavefunction tends to vanish at distances ($r \geq 1$ fm), we will not experience any difficulty in solving the differential equations. It will be shown that r_0 is bigger than 10 fm (for the lowest energy) which implies that the singularity does not cause any problem in the range where QCD applies, on the grounds that the equations have meaning in this particular range.

The χ and Υ states of the “fitting procedure” are described by Eq.(31) and Eqs.((32), (33)), respectively, and these equations can be reduced to a set of twenty, first order differential equations. The problem is a boundary-value problem and two boundary points, $R_{\text{initial}}, R_{\text{final}}$ need to be specified between which the integration will take place. We choose $R_{\text{final}} \simeq 3$ fm where the wavefunction and its derivative are taken to vanish due to the confinement. On the other hand, the choice of R_{initial} requires more care. At small distances, the Coulomb potential dominates the interaction and since the potential is singular at $r = 0$, we are not allowed to start the integration at that point, therefore we choose a point which is much smaller than $r_c \equiv \frac{\alpha_s}{\Lambda}$ (Λ : QCD scale parameter). The QCD scale parameter Λ is not an independent parameter, it is related, instead, to κ [30, 33] through the expression

$$\kappa = \frac{8\pi\Lambda^2}{27} \quad (48)$$

We set $R_{\text{initial}} = 10^{-10}r_c$. The wavefunction behaves as r^γ at $r = R_{\text{initial}}$ while it vanishes at $r = R_{\text{final}}$. According to (34),

$$\gamma = \begin{cases} -1 + \sqrt{j(j+1) - \frac{4}{9}\alpha_s^2} & , \quad \text{for } \chi_{b1} \text{ states} \\ \sqrt{j(j+1) + 1 - \frac{4}{9}\alpha_s^2} & , \quad \text{for } \Upsilon, \chi_{b0}, \chi_{b2} \text{ states} \end{cases} \quad (49)$$

The reason why the bigger value of γ was preferred in the $^3l_{\pm 1}$ states is because the wavefunction goes to zero faster.

The numerical integration of the differential equations is achieved by means of a Runge-

Kutta-based routine and it leads to the following values for the parameters [10]

$$\begin{aligned}
\alpha_s &= 0.36 \pm 0.04 \\
\kappa &= 0.793 \pm 0.008 \frac{\text{GeV}}{\text{fm}} \\
m_b \equiv m &= 4.987 \pm 0.009 \text{ GeV} \\
\lambda_{1S} &\simeq 0.829 (\text{GeV})^2 \\
\lambda_{1P} &\simeq 0.107 (\text{GeV})^2 \\
\lambda_{2P} &\simeq 0.120 (\text{GeV})^2
\end{aligned} \tag{50}$$

which, in turn, give

$$\begin{aligned}
\lambda_{2S} &\stackrel{(41)}{\simeq} 0.147 (\text{GeV})^2 \\
\lambda_{3S} &\stackrel{(42)}{\simeq} 0.126 (\text{GeV})^2 \\
\lambda_{4S} &\stackrel{(43)}{\simeq} 0.072 (\text{GeV})^2 \\
\Lambda &\stackrel{(48)}{=} 409 \pm 40 \text{ MeV}
\end{aligned} \tag{51}$$

Now that the parameters have been estimated, they can be used to calculate the rest of the energy levels of the system. The states that are going to be considered can be classified in three groups according to the set of equations that describe their instantaneous behaviour or, equivalently, the spectroscopic signature: (i). $\eta(1S), \eta(2S), \eta(3S), \eta(4S), h_b(1P), h_b(2P)$, with $^{2S+1}L_J \equiv ^1 l_l$, (ii). $\chi_{b1}(1P), \chi_{b1}(2P)$, with $^{2S+1}L_J \equiv ^3 l_l$, and, finally, (iii). $\Upsilon(1S), \Upsilon(2S), \Upsilon(3S), \Upsilon(4S), \Upsilon(5S), \Upsilon(6S), \chi_{b0}(1P), \chi_{b0}(2P), \chi_{b2}(2P)$, with $^{2S+1}L_J \equiv ^3 l_{l\pm 1}$. By applying Eqs. [(30)-(33)] to the cases (i), (ii) and (iii), and by imposing the same boundary conditions, as previously, the energies of the states can be obtained. The λ s of the wavefunctions entering the equations are dependent on the energies of the corresponding states, therefore the expressions [(44), (45)] should be employed. In addition, Table 1 and Table 2 provide the contribution of the Breit terms to the energy of the system. If all these elements are taken into account, the energy values of the above states are calculated and they are summarised in Table 3 and Table 4. The experimentally established energies are also stated.

The two remaining λ s, namely, $\lambda_{5S}, \lambda_{6S}$ are calculated and they are equal to

$$\begin{aligned}
\lambda_{5S} &\simeq 0.082 (\text{GeV})^2 \\
\lambda_{6S} &\simeq 0.045 (\text{GeV})^2
\end{aligned} \tag{52}$$

Looking at Table 3 and Table 4, there are some points which need to be emphasised:

(i). we notice that the states predicted from the theory have the right accession and almost

the correct energy difference between the P states emerges. Actually, there is not a very big discrepancy between the theoretical results and the experimental data as it can be deduced from Table 5. For higher states, it is getting bigger and the discrepancy becomes larger and this is quite reasonable since those states are characterised by large speed, therefore the v^2 approximation to the Breit terms constitutes a rather rough approximation,

(ii). the part of the energy due to the instantaneous interaction accounts for more of the energy, however, the contribution of terms due to retardation is very important to acquire the correct splitting of the P states. At this point, an interesting feature of the funnel potential should be mentioned. The spin-orbit interaction coming from the Coulomb-like part of the potential (28) differ from the spin-orbit interaction resulting from the linear part (29) in the sign. Thus, if only the Coulomb-like potential was present, it would order the P -levels, in ascending order $^3P_0, ^3P_1, ^3P_0$. On the other hand, the existence solely of the scalar part would order them oppositely. The fact that the previous order is obtained suggests that the spin-orbit part due to the vector potential dominates,

(iii). the ratios $\rho(\chi_{bJ}(1P)), \rho(\chi_{bJ}(2P))$ ($\equiv (2^{++} - 1^{++})/(1^{++} - 0^{++})$) can be calculated from Table 5. They are equal to

$$\begin{aligned}\rho(\chi_{bJ}(1P)) &= 0.69 \pm 0.03 \\ \rho(\chi_{bJ}(2P)) &= 0.56 \pm 0.09\end{aligned}$$

which are in a very good agreement with experiment [41]

$$\begin{aligned}\rho(\chi_{bJ}(1P)) &= 0.67 \pm 0.03 \\ \rho(\chi_{bJ}(2P)) &= 0.58 \pm 0.03\end{aligned}$$

(iv). the states are characterised by small speed ($\langle (v/c)^2 \rangle \leq 0.11$) which make the application of the Pauli approximation and perturbation theory possible. $\langle (v/c)^2 \rangle$ is proportional to the λ of the corresponding state and inversely proportional to the mass of the bottom quark,

(v). r_0 depends on the energy of the state and $r_0 \simeq 12$ fm. $[R_{\text{initial}}, r_0)$ that is clearly larger than the range within which the interaction takes place. If we continue the integration of the differential equations for bigger interparticle distances, we notice that at $r = r_0$ both the wavefunction and its derivative vanish due to confinement.

(vi). the splittings of the $1S, 2S, 3S, 4S$ states are given from Table 6.

The successful description of bottomium by means of the Breit equation can serve as a first indication that the equation can be used in short-range interactions, however, before

we generalise, it would be wise to study the spectrum of other quarkonia composed of lighter quarks.

5.2 The Charmonium system

In this section, a similar procedure is going to be followed in order to calculate the charmonium spectrum. The twelve states we will consider have the quantum numbers $J^{PC} = 1^{--}, 0^{++}, 2^{++}, 0^{-+}, 1^{+-}$. As in the previous subsection, the widths of the decays of the S states into e^+e^- will be employed and the relations between the λ s of the radial part of the wavefunctions will be eventually revealed. The decays lead to ratios given by the following expressions

$$r(2S/1S) \equiv \frac{E_{J/\psi(1S)}^2}{E_{\psi(2S)}^2} \frac{|\psi_{\psi(2S)}(0)|^2}{|\psi_{J/\psi(1S)}(0)|^2} = 0.41 \pm 0.05 \quad (53)$$

$$r(3S/1S) \equiv \frac{E_{J/\psi(1S)}^2}{E_{\psi(3S)}^2} \frac{|\psi_{\psi(3S)}(0)|^2}{|\psi_{J/\psi(1S)}(0)|^2} = 0.14 \pm 0.03 \quad (54)$$

$$r(4S/1S) \equiv \frac{E_{J/\psi(1S)}^2}{E_{\psi(4S)}^2} \frac{|\psi_{\psi(4S)}(0)|^2}{|\psi_{J/\psi(1S)}(0)|^2} = 0.09 \pm 0.02 \quad (55)$$

The parameters that need to be evaluated are three, the mass of the charm quark, $m_c \equiv m$, λ_{1S} and λ_{1P} . It is assumed that the coupling constant α_s and string constant κ are the same as in the bottomium system. This lies in the so-called *flavour independence* [45] of the strong interactions which suggests that the interaction between c, \bar{c} does not differ from that between b, \bar{b} . The same holds true for any combination of two quarks. Since all quarks exist in the same three colour states, they must have identical strong interactions.

The “fitting procedure” [10] is going to be followed, in order to determine the three parameters. We first impose the constraint that the mass of the charm quarks lies within the range $1 \text{ GeV} \leq m \leq 1.6 \text{ GeV}$ (running mass) [41]. The energies of the states $J/\psi(1S)$, $\psi(2S)$, $\chi_{c1}(1P)$ are chosen as input values and the ratios [(53)-(55)] as well as the differential equations Eqs. [(31)-(33)] are used. The latter are integrated in the range $[10^{-10}r_c, 3 \text{ fm}]$ where $r_c = \alpha_s/\Lambda$. By solving the differential equations and by taking into account the relations

between the parameters imposed by [(53)-(55)], we obtain the following results [10]

$$\begin{aligned} m_c &\equiv = 1.572 \pm 0.009 \text{ GeV} \\ \lambda_{1S} &\simeq 0.395 \text{ (GeV)}^2 \\ \lambda_{1P} &\simeq 0.074 \text{ (GeV)}^2 \end{aligned} \tag{56}$$

which, in turn, give

$$\begin{aligned} \lambda_{2S} &\stackrel{(53)}{\simeq} 0.083 \text{ (GeV)}^2 \\ \lambda_{3S} &\stackrel{(54)}{\simeq} 0.040 \text{ (GeV)}^2 \\ \lambda_{4S} &\stackrel{(55)}{\simeq} 0.030 \text{ (GeV)}^2 \end{aligned} \tag{57}$$

Now that the parameters have been estimated, they can be used to calculate the rest of the energy levels of the system. The application of Eqs.[(30)-(33)] and Table 7 lead to the results summarised in Table 8. These values can be used to calculate the S and P splittings which have been measured in the laboratory and their discrepancy and, eventually, the success of the theory can be evaluated (Table 9, Table 10). We notice there is a discrepancy of 18% between the theoretical and the experimental splitting of the $\chi_{c2}(1P)$ and $\chi_{c1}(1P)$ states. The ratio $\rho = (2^{++} - 1^{++}) / (1^{++} - 0^{++})$ equals $\rho = 0.420 \pm 0.005$ which is well below the experimentally calculated ($\equiv 0.49$) [41]. Thus, although, some of the energy levels seem to be in a very good agreement with experiment, the results should be taken with great care. The reason why the charmonium system is not described entirely adequately by the theoretical model is that our theory considers the static interaction as the main contributor to the system's energy and it treats the retarded terms perturbatively. The problem is that for charmonium, this scheme does not provide exactly the correct energy as it nearly does for bottomium, resulting to larger deviations from experimental values. In contrast to bottomium where the corrections due to the Breit terms constitute a small proportion of the total energy, in charmonium they are quite large. Anyway, larger corrections to the static energy (which, also, contains relativistic contributions) were expected since these terms are inversely proportional to the mass of the constituent particles. By looking at the expectation value of the square of the speed of the systems (Table 3, Table 4 and Table 8), we can conclude that, indeed, although the relativistic corrections are not negligible, to some extent, both bottomium and charmonium can be regarded as non-relativistic systems, with this assertion suiting more to bottomium.

Another reason for the failure to obtain the correct results is the form of the potential that is supposed to describe strong interactions. Although the potential employed exhibits the correct behaviour at small and large distances, we are not able to determine what it looks like in between. In other words, we do not know its exact form. Certainly, a knowledge of this would lead to better results.

In addition, we should not forget the order to which the energy values has been calculated. If the wavefunction used, corresponded to the solutions of the Coulomb problem, the spectrum would contain corrections of order up to α_s^4 . For the wavefunctions of the harmonic oscillator, the energies contain terms of order up to α_s . If we wish to calculate the energies more accurately, we need to consider higher order terms since their inclusion would provide an essential contribution to the energy levels, however, again, the results should not agree exactly with experiment. This is because it is not possible to take into account all diagrams. Thus, we should better compare our results with other theories. We consider the study conducted by Eichten *et al.* (they, also, used a funnel potential as a possible form of the interaction) [28] on charmonium. They fitted the parameters ($\alpha_s = 0.39$, $\kappa = 926$ GeV/fm) and calculated the energies summarised in Table 11, Table 12. The asterisk (*) refers to the states used to fit the parameters. Their values are slightly different from those we used because at the time those calculations were carried out, the experimental data differed from the recent results. The P - states, also, in these tables represent the centre-of-mass state of the $j = 0$, $j = 1$, $j = 2$ states.

The coupling constant of the interaction and the QCD scale parameter were calculated in the previous subsection where we studied the bottomium spectrum. One might wonder whether it would be wise to allow the values of these parameters to be derived from the study of charmonium, instead. The problem is that in order to estimate these values, one approximation has been made already by optimising the radial wavefunctions of the three-dimensional harmonic oscillator (in addition to the approximation concerning the form of the potential and the approximation related to the effective interaction). A more rigorous and accurate procedure should assume the use of the wavefunctions of the funnel potential. It is true that charmonium exhibits a more relativistic behaviour than bottomium and a very careful consideration should be taken, otherwise there is a danger of obtaining unreliable results. The fact that bottomium consists of heavier particles and thereby the

system is regarded as less relativistic implies that such approximations as those performed throughout this work are less likely to result in unacceptable and meaningless results if the parameters are calculated for bottomium.

6 Conclusions

In this paper, the application of the Breit equation to bound state of systems characterised by short-range interactions was studied. Despite the clear lack of covariance of the equation, we assumed that it can be considered to be approximately Lorentz invariant for systems interacting through a very strong short-range potentials such as quarkonia and more specifically, bottomium and charmonium. A funnel potential was introduced as a candidate potential describing QCD and the Pauli approximation of the Breit equation was employed in order to distinguish the contribution of the static terms of the interaction from that part of the interaction responsible for retardation. The solution of the Breit equation helped to obtain the spectrum of bottomium and charmonium and to compare them with experimental data. The S and P states as well as their splitting were calculated and it turned out that the results are, in general, in a very good agreement with experiment. The Breit equation was, also, used to determine the coupling constant and the scale parameter of the strong interaction and they were estimated to be $\alpha_s = 0.36 \pm 0.04$ and $\Lambda = 409 \pm 40\text{GeV}$, respectively.

A part of the success was attributed to the fact that the small speed of the states allowed the Breit terms to be treated perturbatively despite the relatively large coupling constant (α_s/m^2 is small while α_s is large). The large mass of the constituent particles makes the systems less relativistic, therefore the static part of the interaction contributes much more to the binding energy than the Breit terms.

Unfortunately, for systems that are characterised by larger speed or, equivalently, by a smaller mass, the discrepancy between experiment and the theory is large enough to raise concerns over the reliability of the Breit equation as a satisfactory theory. This suggests that, in that case, the Breit equation is not adequate to describe systems of two fermions and an alternative equation should be employed.

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| Meson States | Energy due to the Static potential | Energy corrections due to the Breit terms $\left(A_{\text{index}} = \frac{1}{9} \frac{\alpha_s}{m_b^2} \sqrt{\frac{\lambda_{\text{index}}^3}{\pi}}\right)$ |
|-----------------|------------------------------------|--|
| $\Upsilon(1S)$ | $E_{\text{static}}(\Upsilon(1S))$ | $+8.0A_{1S}$ |
| $\Upsilon(2S)$ | $E_{\text{static}}(\Upsilon(2S))$ | $-80.0A_{2S}$ |
| $\Upsilon(3S)$ | $E_{\text{static}}(\Upsilon(3S))$ | $-183.2A_{3S}$ |
| $\Upsilon(4S)$ | $E_{\text{static}}(\Upsilon(4S))$ | $-279.7A_{4S}$ |
| $\Upsilon(5S)$ | $E_{\text{static}}(\Upsilon(5S))$ | $-371.0A_{5S}$ |
| $\Upsilon(6S)$ | $E_{\text{static}}(\Upsilon(6S))$ | $-458.1A_{6S}$ |
| $\chi_{b0}(1P)$ | $E_{\text{static}}(\chi_{b0}(1P))$ | $-320.0A_{1P}$ |
| $\chi_{b1}(1P)$ | $E_{\text{static}}(\chi_{b1}(1P))$ | $-160.0A_{1P}$ |
| $\chi_{b2}(1P)$ | $E_{\text{static}}(\chi_{b2}(1P))$ | $-70.4A_{1P}$ |
| $\chi_{b0}(2P)$ | $E_{\text{static}}(\chi_{b0}(2P))$ | $-518.4A_{2P}$ |
| $\chi_{b1}(2P)$ | $E_{\text{static}}(\chi_{b1}(2P))$ | $-310.4A_{2P}$ |
| $\chi_{b2}(2P)$ | $E_{\text{static}}(\chi_{b2}(2P))$ | $-193.9A_{2P}$ |

Table 1: Energy levels of bottomium as a function of the parameters α_s, m_b, λ_s .

| Meson States | J^{PC} | Energy due to the Static potential | Energy corrections due to the Breit terms $\left(A_{\text{index}} = \frac{1}{9} \frac{\alpha_s}{m_b^2} \sqrt{\frac{\lambda_{\text{index}}^3}{\pi}}\right)$ |
|--------------|----------|------------------------------------|--|
| $\eta(1S)$ | 0^{-+} | $E_{\text{static}}(\eta(1S))$ | $-48.0A_{1S}$ |
| $\eta(2S)$ | 0^{-+} | $E_{\text{static}}(\eta(2S))$ | $-416.0A_{2S}$ |
| $\eta(3S)$ | 0^{-+} | $E_{\text{static}}(\eta(3S))$ | $-603.2A_{3S}$ |
| $\eta(4S)$ | 0^{-+} | $E_{\text{static}}(\eta(4S))$ | $-769.7A_{4S}$ |
| $h(1P)$ | 1^{+-} | $E_{\text{static}}(h(1P))$ | $-128.0A_{1P}$ |
| $h(2P)$ | 1^{+-} | $E_{\text{static}}(h(2P))$ | $-268.8A_{2P}$ |

Table 2: Energy levels of not experimentally measured bottomimum states as a function of the parameters α_s, m_b, λ_s .

| Meson States | J^{PC} | Energy (Experim.) (MeV) | Energy (Theory)(MeV) | | $\left\langle \left(\frac{v}{c}\right)^2 \right\rangle$ |
|------------------------|----------|-------------------------|-----------------------|---------------------|---|
| | | | Static part (MeV) | Retarded part (MeV) | |
| $\Upsilon(1S)$ 2nd | 1^{--} | 9460.37 ± 0.21 | $9460.37 \pm 0.21^*$ | | 0.050 |
| | | | 9454.89 ± 0.21 | $+5.48 \pm 0.01$ | |
| $\Upsilon(2S)$ 8th | 1^{--} | 10023.30 ± 0.31 | $10023.30 \pm 0.31^*$ | | 0.083 |
| | | | 10027.40 ± 0.31 | -4.101 ± 0.007 | |
| $\Upsilon(3S)$ 14th | 1^{--} | 10355.3 ± 0.5 | $10355.3 \pm 0.5^*$ | | 0.111 |
| | | | 10362.9 ± 0.5 | -7.42 ± 0.01 | |
| $\Upsilon(4S)$ 16th | 1^{--} | 10580.0 ± 3.5 | $10580.0 \pm 3.5^*$ | | 0.087 |
| | | | 10584.9 ± 3.5 | -4.929 ± 0.008 | |
| $\Upsilon(5S)$ 17th | 1^{--} | 10865 ± 8 | 10882 ± 8 | | 0.125 |
| | | | 10890 ± 8 | -7.86 ± 0.01 | |
| $\Upsilon(6S)$ 18th | 1^{--} | 11019 ± 8 | 11037 ± 9 | | 0.083 |
| | | | 11041 ± 9 | -3.944 ± 0.007 | |
| $\eta(1S)$ 1st | 0^{-+} | | 9393.9 ± 0.3 | | 0.050 |
| | | | 9426.8 ± 0.3 | -32.88 ± 0.08 | |
| $\eta(2S)$ 7th | 0^{-+} | | 9973.3 ± 0.4 | | 0.083 |
| | | | 9994.6 ± 0.4 | -21.32 ± 0.04 | |
| $\eta(3S)$ 13th | 0^{-+} | | 10324 ± 2 | | 0.111 |
| | | | 10348 ± 2 | -24.42 ± 0.04 | |
| $\eta(4S)$ 15th | 0^{-+} | | 10553 ± 1 | | 0.087 |
| | | | 10567 ± 1 | -13.56 ± 0.02 | |
| $\chi_{b0}(1P)$ 3rd | 0^{++} | 9859.8 ± 1.3 | 9858.9 ± 0.4 | | 0.043 |
| | | | 9869.1 ± 0.4 | -10.15 ± 0.02 | |
| $\chi_{b1}(1P)$ 5th | 1^{++} | 9891.9 ± 0.7 | $9891.9 \pm 0.7^*$ | | 0.043 |
| | | | 9897.0 ± 0.7 | -5.077 ± 0.001 | |
| $\chi_{b2}(1P)$ 6th | 2^{++} | 9913.2 ± 0.6 | 9914.7 ± 0.9 | | 0.043 |
| | | | 9916.9 ± 0.9 | -2.234 ± 0.004 | |
| $h(1P)$ 4th | 1^{+-} | | 9906 ± 1 | | 0.043 |
| | | | 9910 ± 1 | -4.062 ± 0.007 | |

Table 3: The energy spectrum of the bottomium system derived from experiment (third column), the Breit equation (fourth and fifth columns) and the expectation values of the square of the velocity operator v for each state (sixth column). The asterisk (*) refers to the states used to fit the parameters. The position of every state in the spectrum (for the states we have considered) is mentioned in the first column.

| Meson States | J^{PC} | Energy (Experi.m.) (MeV) | Energy (Theory)(MeV) | | $\left\langle \left(\frac{v}{c}\right)^2 \right\rangle$ |
|-------------------------|----------|--------------------------------|----------------------|------------------------|---|
| | | | Static part (MeV) | Retarded part (MeV) | |
| $\chi_{b0}(2P)$ 9th | 0^{++} | 10232.1 ± 0.6 | 10234.3 ± 0.2 | | 0.087 |
| | | | 10253.9 ± 0.2 | -19.60 ± 0.03 | |
| $\chi_{b1}(2P)$ 11th | 1^{++} | 10255.2 ± 0.5 | $10255.2 \pm 0.5^*$ | | 0.087 |
| | | | 10266.9 ± 0.5 | -11.74 ± 0.02 | |
| $\chi_{b2}(2P)$ 12th | 2^{++} | 10268.5 ± 0.4 | 10266.9 ± 0.7 | | 0.087 |
| | | | 10274.3 ± 0.7 | -7.33 ± 0.01 | |
| $h(2P)$ 10th | 1^{+-} | | 10260.2 ± 0.9 | | 0.087 |
| | | | 10270.4 ± 0.9 | -10.17 ± 0.02 | |

Table 4: The energy spectrum of the bottomium system derived from experiment (third column), the Breit equation (fourth and fifth columns) and the expectation values of the square of the velocity operator v for each state (sixth column). The asterisk (*) refers to the states used to fit the parameters. The position of every state in the spectrum (for the states we have considered) is mentioned in the first column.

| ΔM | Theory (MeV) | Experiment (MeV) | Deviation (%) |
|---------------------------------------|----------------|------------------|---------------|
| $m(\chi_{b1}(1P)) - m(\chi_{b0}(1P))$ | 33.0 ± 0.8 | 32 ± 1 | 3.1 |
| $m(\chi_{b2}(1P)) - m(\chi_{b1}(1P))$ | 22.8 ± 1.1 | 21.3 ± 0.9 | 7.0 |
| $m(\chi_{b1}(2P)) - m(\chi_{b0}(2P))$ | 20.9 ± 0.8 | 23.1 ± 0.8 | 9.5 |
| $m(\chi_{b2}(2P)) - m(\chi_{b1}(2P))$ | 11.7 ± 0.9 | 13.3 ± 0.6 | 12.0 |

Table 5: Splittings of the $1P$ and $2P$ levels of bottomium.

| ΔM | Theory (MeV) |
|---------------------------------|----------------|
| $m(\Upsilon(1S)) - m(\eta(1S))$ | 66.5 ± 0.4 |
| $m(\Upsilon(2S)) - m(\eta(2S))$ | 50.0 ± 0.5 |
| $m(\Upsilon(3S)) - m(\eta(3S))$ | 31 ± 2 |
| $m(\Upsilon(4S)) - m(\eta(4S))$ | 27 ± 4 |

Table 6: Splittings of the S levels of bottomium.

| Meson States | Energy due to the Static potential | Energy corrections due to the Breit terms $\left(A_{\text{index}} = \frac{1}{9} \frac{\alpha_s}{m_c^2} \sqrt{\frac{\lambda_{\text{index}}^3}{\pi}} \right)$ |
|-----------------|------------------------------------|--|
| $J/\psi(1S)$ | $E_{\text{static}}(J/\psi(1S))$ | $+8.0A_{1S}$ |
| $\psi(2S)$ | $E_{\text{static}}(\psi(2S))$ | $-80.0A_{2S}$ |
| $\psi(3S)$ | $E_{\text{static}}(\psi(3S))$ | $-183.2A_{3S}$ |
| $\psi(4S)$ | $E_{\text{static}}(\psi(4S))$ | $-279.7A_{4S}$ |
| $\eta_c(1S)$ | $E_{\text{static}}(\eta(1S))$ | $-48.0A_{1S}$ |
| $\eta_c(2S)$ | $E_{\text{static}}(\eta(2S))$ | $-416.0A_{2S}$ |
| $\eta_c(3S)$ | $E_{\text{static}}(\eta(3S))$ | $-603.2A_{3S}$ |
| $\eta_c(4S)$ | $E_{\text{static}}(\eta(4S))$ | $-769.7A_{4S}$ |
| $h_c(1P)$ | $E_{\text{static}}(h_c(1S))$ | $-128.0A_{1S}$ |
| $\chi_{c0}(1P)$ | $E_{\text{static}}(\chi_{c0}(1P))$ | $-320.0A_{1P}$ |
| $\chi_{c1}(1P)$ | $E_{\text{static}}(\chi_{c1}(1P))$ | $-160.0A_{1P}$ |
| $\chi_{c2}(1P)$ | $E_{\text{static}}(\chi_{c2}(1P))$ | $-70.4A_{1P}$ |
| $\chi_{c1}(1P)$ | $E_{\text{static}}(\chi_{c1}(1P))$ | $-160.0A_{1P}$ |
| $\chi_{c2}(1P)$ | $E_{\text{static}}(\chi_{c2}(1P))$ | $-70.4A_{1P}$ |

Table 7: Energy levels of charmonium as a function of the parameters $\alpha_s, m_c, \mu, \lambda_s$.

| Meson States | J^{PC} | Energy (Experim.) (MeV) | Energy (Theory)(MeV) | | $\left\langle \left(\frac{v}{c}\right)^2 \right\rangle$ |
|------------------------|----------|-------------------------------|----------------------|------------------------|---|
| | | | Static part (MeV) | Retarded part (MeV) | |
| $J/\psi(1S)$ 2nd | 1^{--} | 3096.88 ± 0.04 | 3096.88 ± 0.04 * | | 0.24 |
| | | | 3078.68 ± 0.04 | $+18.2 \pm 0.1$ | |
| $\psi(2S)$ 8th | 1^{--} | 3686.00 ± 0.09 | 3686.00 ± 0.09 * | | 0.47 |
| | | | 3703.58 ± 0.08 | -17.58 ± 0.03 | |
| $\psi(3S)$ 10th | 1^{--} | 4040 ± 10 | 4106 ± 10 | | 0.36 |
| | | | 4120 ± 10 | -13.54 ± 0.02 | |
| $\psi(4S)$ 12th | 1^{--} | 4415 ± 6 | 4454 ± 8 | | 0.37 |
| | | | 4467 ± 8 | -13.44 ± 0.02 | |
| $\eta(1S)$ 1st | 0^{-+} | 2979.8 ± 2.1 | 2987 ± 2 | | 0.24 |
| | | | 3095 ± 2 | -108.0 ± 0.8 | |
| $\eta(2S)$ 7th | 0^{-+} | 3594 ± 5 | 3601 ± 1 | | 0.47 |
| | | | 3692 ± 1 | -91.4 ± 0.2 | |
| $\eta(3S)$ 9th | 0^{-+} | | 4023 ± 1 | | 0.36 |
| | | | 4068 ± 1 | -44.59 ± 0.08 | |
| $\eta(4S)$ 11th | 0^{-+} | | 4379.1 ± 0.9 | | 0.37 |
| | | | 4416.1 ± 0.9 | -36.98 ± 0.07 | |
| $\chi_{c0}(1P)$ 3rd | 0^{++} | 3417.3 ± 2.8 | 3422 ± 1 | | 0.30 |
| | | | 3481 ± 1 | -59.0 ± 0.1 | |
| $\chi_{c1}(1P)$ 5th | 1^{++} | 3510.53 ± 0.12 | 3510.53 ± 0.12 * | | 0.30 |
| | | | 3540.02 ± 0.12 | -29.49 ± 0.05 | |
| $\chi_{c2}(1P)$ 6th | 2^{++} | 3556.17 ± 0.13 | 3547.9 ± 0.2 | | 0.30 |
| | | | 3560.9 ± 0.2 | -12.98 ± 0.02 | |
| $h(1P)$ 4th | 1^{+-} | 3526.14 ± 0.24 | 3483 ± 2 | | 0.30 |
| | | | 3507 ± 2 | -23.60 ± 0.04 | |

Table 8: The energy spectrum of the charmonium system derived from the experiment (third column), the Breit equation (fourth and fifth columns) and the expectation values of the square of the velocity operator v for each state (sixth column). The asterisk (*) refers to the states used to fit the parameters. The position of every state in the spectrum (for the states we have considered) is mentioned in the first column.

| ΔM | Theory (MeV) | Experiment (MeV) | Deviation (%) |
|-------------------------------|--------------|------------------|---------------|
| $m(\psi(1S)) - m(\eta_c(1S))$ | 110 ± 2 | 117 ± 2 | 6.0 |
| $m(\psi(2S)) - m(\eta_c(2S))$ | 85 ± 1 | 92 ± 5 | 7.6 |
| $m(\psi(3S)) - m(\eta_c(3S))$ | 83 ± 10 | | |
| $m(\psi(4S)) - m(\eta_c(4S))$ | 75 ± 8 | | |

Table 9: Splittings of the S levels of charmonium.

| ΔM | Theory (MeV) | Experiment (MeV) | Deviation (%) |
|---------------------------------------|----------------|------------------|---------------|
| $m(\chi_{c1}(1P)) - m(\chi_{c0}(1P))$ | 89 ± 1 | 96 ± 1 | 4.3 |
| $m(\chi_{c2}(1P)) - m(\chi_{c1}(1P))$ | 37.4 ± 0.2 | 45.6 ± 0.2 | 18.0 |

Table 10: Splittings of the $1P$ states of charmonium.

| Charmonium $m_c = 1.84$ GeV | |
|-----------------------------|--------------|
| State | Energy (MeV) |
| $J/\psi(1S)$ | 3095* |
| $\psi(2S)$ | 3684* |
| $\psi(3S)$ | 4110 |
| $\psi(4S)$ | 4460 |
| $\psi(5S)$ | 4790 |
| $\chi_c(1P)$ | 3522* |

Table 11: The charmonium spectrum according to Eichten *et al.* [28]

| Bottomium $m_b = 5.17$ GeV | |
|----------------------------|--------------|
| State | Energy (MeV) |
| $\Upsilon(1S)$ | 9460* |
| $\Upsilon(2S)$ | 10050 |
| $\Upsilon(3S)$ | 10400 |
| $\Upsilon(4S)$ | 10670 |
| $\Upsilon(5S)$ | 10920 |
| $\Upsilon(6S)$ | 11140 |
| $\chi_b(1P)$ | 9960 |
| $\chi_b(2P)$ | 10310 |
| $\chi_b(3P)$ | 10600 |

Table 12: The bottomium spectrum according to Eichten *et al.* [28]